The Cauchy–Poisson problem for a rotating liquid

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A semi-infinite body of liquid is in uniform rotation about a vertical axis for $t < 0$. A concentrated, vertical displacement of the free surface is imposed at $t = 0$. The motion of the free surface for $t > 0$ is calculated in linear approximation with the aid of Hankel and Laplace transformations, together with an intermediate transformation to parabolic co-ordinates. The result appears as an integral superposition of dispersive waves that divides naturally into two parts, corresponding to waves of the first and second class, with angular frequencies that are respectively greater or less than twice the angular speed of rotation. The waves of the first class, which are qualitatively similar to those in the classical (Cauchy–Poisson) problem, are found to dominate the asymptotic representation, as obtained through a stationary-phase approximation. The analysis is carried out in such a way as to separate the effects of Coriolis acceleration and free-surface curvature. Attention is focused on concave surfaces, such as would be realized in a laboratory experiment, but it is pointed out that striking differences exist between the dispersion laws for concave and convex surfaces, especially as regards waves of the second class.

1. Introduction

The study of surface waves in a rotating liquid was initiated by Kelvin (1879), and continued by others (Lamb 1932, §§ 207–212), on the basis of shallow-water theory and the approximation of the free surface by a plane (*planar approximation*). Miles (1959) extended this earlier work by eschewing the shallow-water approximation but retained the planar approximation. A significant feature of these investigations, in so far as simple harmonic oscillations are assumed to exist, is a degeneration of the solutions at, and some degree of non-uniform validity in the neighbourhood of, $\sigma = 2\omega$ (σ denotes angular frequency of oscillation and ω angular velocity of rotation). It seems likely that this difficulty arises from the *a priori* assumption of a steady state, although the planar approximation also introduces an element of uncertainty. It therefore appears desirable to consider the effects of rotation in a consistently posed initial-value problem as $t \to \infty$.

[Non-free-surface problems in a rotating fluid have been surveyed by Squire (1956). Sretenskii (1960) and Lauwerier (1961) have considered transient surface waves in a rotating liquid on the joint basis of the shallow-water and planar approximations.]

Motivated by these considerations, we shall consider here the response to an initially concentrated displacement of the paraboloidal free surface

$$z = z_0(r) = r^2/2l, \tag{1.1}$$

bounding a semi-infinite, rotating body of liquid. Such a configuration could be approximated, as regards observations remote from the actual boundaries, in a laboratory tank; equilibrium between the uniform gravitational field and the centrifugal force at $z = z_0$ then would imply $l = g/\omega^2$. More generally, we may refer our investigation to the non-uniform gravitational field described by the potential

$$\psi = \tfrac{1}{2}\omega^2 r^2 + g[z - z_0(r)] \tag{1.2}$$

and characterized by the dimensionless parameter

$$\alpha = \omega^2 l/g. \tag{1.3}$$

In so far as $\alpha \geqslant 0$, this generalization will be found to entail no essential increase in the complexity of the analysis, compared with that required for the aforementioned laboratory problem (for which $\alpha = 1$), and will allow us to separate the effects of Coriolis acceleration and free-surface curvature. We remark that the neglect of the latter effect, as in the planar approximation, is clearly appropriate for $\alpha \gg 1$; on the other hand, it is definitely inconsistent for axisymmetric motion if $\alpha = O(1)$.

[The Coriolis-acceleration-induced frequency shift of natural oscillations in a finite domain is $O(\omega)$ for asymmetric, and $O(\omega^2)$ for axisymmetric, oscillations as $\omega \to 0$; the free-surface slope enters the kinematic boundary condition linearly and induces a frequency shift of $O(\omega^2/\alpha)$. We infer from these considerations that the planar approximation *is* appropriate to the study of asymmetric waves in a slowly rotating fluid; such conditions obtain in the oceanographic problems that motivated Kelvin and his successors. The fact that free-surface curvature may not be neglected in a consistent treatment of axisymmetric oscillations appears to have escaped the notice of these earlier investigators and to have been first pointed out by Fultz (1962), whose experimental investigations yielded frequencies consistently lower than those predicted by the earlier analyses.]

In the absence of rotation and free-surface curvature, the problem posed in the penultimate paragraph reduces to the classical Cauchy–Poisson problem (Lamb 1932, § 255). This reduced problem is distinguished by the absence of characteristic scales for either length or time, in consequence of which the solution for the free-surface displacement in the radial co-ordinate r and the time t has the similarity form

$$\zeta(r,t) = Ar^{-2}f_0(gt^2/r). \tag{1.4}$$

It is possible to obtain this solution by posing the velocity potential as a function of the similarity variable $\mu = gt^2/r$ and the spherical polar angle θ; separation of variables then yields a power-series representation of $f_0(\omega)$. It is, however, generally more enlightening to pose the solution as an integral superposition of surface waves that: (*a*) have the form

$$\zeta_k(r,t) = J_0(kr)\cos(\sigma t) \tag{1.5}$$

on the free surface $z = 0$; (*b*) fall off like $\exp(-k|z|)$ with depth; and (*c*) are governed by the simple dispersion law

$$\sigma^2 = gk. \tag{1.6}$$

Such a representation may be developed as either a power or an asymptotic series in u, of which the leading terms are given by

$$f_0(\mu) = \tfrac{1}{2}\mu + O(\mu^3) \qquad (\mu \to 0) \tag{1.7}$$

and

$$f_0(\mu) \sim 2^{-\frac{3}{2}}\mu \cos(\tfrac{1}{4}\mu) \quad (\mu \to \infty), \tag{1.8}$$

provided that we define $2\pi A$ as the initially displaced volume.

We may expect rotation to alter these results, both quantitatively and qualitatively, through the more or less distinct effects of Coriolis acceleration and free-surface curvature. These effects carry with them the time scale $1/\omega$ and the length scale l ($2l$ is the latus rectum of the paraboloid of (1.1)), in consequence of which a similarity solution no longer exists. However, the representation of the free-surface displacement in terms of the elementary oscillations of (1.5) depends only on axial symmetry and linearity, and we therefore may construct a solution by an appropriate generalization of the classical procedure.

Following Lamb (§ 223), we may divide oscillations in a rotating liquid into 'waves of the first class' and 'waves of the second class' according to whether $|\sigma| > 2\omega$ or $|\sigma| < 2\omega$. Waves of the first class are qualitatively similar to classical gravity waves in that they decay exponentially in a fluid of infinite depth and are governed by a unique dispersion law (in the sense that there exists at most one value of σ^2 for each value of k).

Waves of the second class owe their existence entirely to rotation and have no antecedents in a non-rotating liquid (at least in the present problem; for some configurations they degenerate to steady motions as $\omega \to 0$). Strictly speaking, they are not *surface* waves, in that they propagate into the liquid without decay (other than that associated with radial spreading). An even more important distinction, at least from an analytical point of view, is that an infinite number of frequencies generally is admissible for a given wave-number. As we shall show, the frequency spectrum for a concave free surface is continuous over $(-2\omega, 2\omega)$; that for a convex surface is, in remarkable contrast, discrete but infinite, with finite gaps inside $\pm 2\omega$ and a limit point at $\sigma = 0$.

This division of waves into two classes suggests that we generalize (1.1) according to

$$\zeta(r,t) = Ar^{-2}f, \quad f = f_1(\mu, \theta; \alpha) + f_2(\tau, \theta; \alpha), \tag{1.9 a, b}$$

where $2\pi A$ is the initially displaced volume, f_1 and f_2 represent waves of the first and second class, and

$$\mu = gt^2/r, \quad \tau = \omega t, \quad \theta = r/l. \tag{1.10 a, b, c}$$

We remark that θ is simply the radial slope of the equilibrium free surface. A more suitable representative of r for $|\alpha| \gg 1$ would be

$$\alpha\theta = \omega^2 r/g, \tag{1.11}$$

which is independent of l. In particular, we must let $|\alpha| \to \infty$ with $\alpha\theta$ fixed in order to reduce our results to those that otherwise might be obtained by an *a priori* invocation of the planar approximation.

The details of our analysis, which involves an intermediate excursion into parabolic co-ordinates, are rather involved and add little to the physical interpretation of the results; accordingly, we shall present and discuss the more important results at this point. The central result, obtained in § 3, is the Hankel–Laplace transform $Z(k, i\sigma)$, which gives the joint distribution of the elementary free-surface displacements of (1.5) over the wave-number (k) and frequency (σ) spectra. Noting the initial behaviour [cf. (1.7)]

$$f = \tfrac{1}{2}\mu[1+O(\theta^2)] \quad (\mu \to 0,\ \theta \to 0) \tag{1.12}$$

only in passing, we shall separate the σ-distribution into waves of the first and second class and then apply standard techniques to obtain the approximations

$$f_1 \sim 2^{-\frac{3}{2}}[\mu - 12\alpha\theta - \theta + O(\mu^0, \theta^2\mu^{-1})] \times \cos[\tfrac{1}{4}\mu + 2\alpha\theta + \tfrac{1}{2}\theta + O(\mu^{-1}, \theta^2\mu^{-1})] \quad (\mu \gg 1,\ \theta,\ \alpha\theta) \tag{1.13}$$

and

$$f_2 = 2\alpha\theta[\tau^{-1}J_1(2\tau) + O(\theta)] \quad (\theta \to 0) \tag{1.14a}$$

$$= 2\pi^{-\frac{1}{2}}\alpha\theta\tau^{-\frac{3}{2}}[\sin(2\tau - \tfrac{1}{4}\pi) + O(\theta^{\frac{1}{2}}\tau^{\frac{1}{2}})] \quad (\tau \to \infty). \tag{1.14b}$$

Comparing (1.13) and (1.14), we find that the waves of the second class are dominated by those of the first class for sufficiently large values of μ. This dominance extends to the effects of rotation—i.e. to the terms of $O(\theta, \alpha\theta)$—only for $\tau \gg 1$; however, the equality

$$\tau = (\alpha\theta\mu)^{\frac{1}{2}} \tag{1.15}$$

implies $\alpha\theta = O(1/\mu)$ if $\tau = O(1)$, and the terms of $O(\theta, \alpha\theta)$ in (1.13) then are of the same order as terms already neglected. We therefore may neglect f_2 compared with f_1 in a consistent, asymptotic approximation to the free-surface displacement. Substituting (1.13) into (1.9), we then obtain

$$\zeta(r,t) \sim \frac{2^{-\frac{3}{2}}A}{r^2}\left[\frac{gt^2}{r} - \left(12+\frac{1}{\alpha}\right)\frac{\omega^2 r}{g}\right]\cos\left(\left|\frac{gt^2}{4r}\right| + \epsilon\right) \quad (t \to \infty), \tag{1.16}$$

where

$$\epsilon = \left(2+\frac{1}{\alpha}\right)\frac{\omega^2 r}{g}. \tag{1.17}$$

We conclude that the principal effect of rotation on the Cauchy–Poisson problem is the phase shift ϵ in the asymptotic wave train; the change in amplitude, although of the same order of magnitude as the phase shift, has a uniformly small effect. All other effects are evanescent as $t \to \infty$. Finally, we remark that invoking the planar approximation by replacing $\alpha = 1$ (as for the laboratory problem) by $\alpha = \infty$ in (1.17) leads to a 20 % error in the phase-shift ϵ.

It might be thought that our investigation also should be applicable to geophysical problems, where the curvature of the free surface, approximated locally by a paraboloid, is convex and the parameter α is small ($|\alpha| \doteqdot \frac{1}{289}\sin^2\beta$ if $-l$ is taken to be the radius of the Earth and β the latitude). In fact, the assumption of infinite depth renders our results rather artificial for such applications (especially for oceanographic problems, where the shallow-water approximation is rather more appropriate), and we therefore shall not attempt to render our analysis generally valid for convex free surfaces. Nevertheless, some of the differences between waves in rotating liquids with convex and concave free surfaces are rather striking, and we have thought it worth while to sketch (in § 6 below) the modifications required to render our analysis valid for $\alpha < 0$ and then to state the essential, qualitative differences in the corresponding results.†

2. Formulation

We consider a semi-infinite body of liquid, bounded by

$$-\infty < z \leqslant z_0(r) \quad (0 \leqslant r < \infty) \tag{2.1}$$

in cylindrical polar co-ordinates, to be in uniform rotation with angular velocity ω about the vertical axis $r = 0$. We require the disturbed free surface,

$$z = z_0(r) + \zeta(r, t),$$

subsequent to an initial elevation

$$z - z_0 = \zeta_0(r) = A\delta(r)/r \quad (t = 0), \tag{2.2}$$

where $2\pi A$ is the displaced volume associated with ζ_0 and $\delta(r)$ the Dirac delta function.

We shall assume that ζ_0 is sufficiently small to justify the linearization of both the equations of motion and the free-surface boundary conditions. Invoking axial symmetry, we then may determine the perturbation pressure p and the velocity $\{u, v, w\}$ in a rotating (with the angular velocity ω) reference frame‡ from an acceleration potential $\chi(r, z, t)$ [cf. (1.2)] according to

$$p/\rho + g(z - z_0) = \chi, \tag{2.3}$$

$$u_t - 2\omega v = -\chi_r, \tag{2.4a}$$

$$v_t + 2\omega u = 0, \tag{2.4b}$$

$$w_t = -\chi_z, \tag{2.4c}$$

and

$$\nabla^2 \chi_{tt} + 4\omega^2 \chi_{zz} = 0, \tag{2.5}$$

where ρ denotes the density of the liquid and subscripts imply partial differentiation. The free-surface boundary conditions are

$$p = 0, \quad w = D\zeta/Dt \quad (z = z_0 + \zeta \doteqdot z_0) \tag{2.6a, b}$$

or, equivalently,

$$\chi = g\zeta, \quad w = \zeta_t + uz_0'(r) \quad (z = z_0). \tag{2.7a, b}$$

The initial conditions are

$$\chi = \chi_t = 0 \quad (t = 0, z < z_0) \tag{2.8}$$

and

$$\zeta = \zeta_0(r), \quad \zeta_t = 0 \quad (t = 0). \tag{2.9}$$

We also must require χ and its derivatives to vanish as r and/or $-z \to \infty$.

† I am indebted to the referee for suggesting that significant differences might exist and also for suggesting that the importance of free-surface curvature, relative to that of Coriolis acceleration, might be established by comparing the results obtained (for $\alpha = 1$) with and without the planar approximation.

‡ The cylindrical polar angle is not required in this reference frame by virtue of the axial symmetry, but there is a tangential component of the perturbation velocity in consequence of the Coriolis acceleration.

A more complete derivation of, and discussion of the approximations antecedent to, (2.3)–(2.6) is given in Miles (1959); however, the term uz_0' in (2.7 *b*) was neglected in this earlier paper in consequence of the planar approximation.

3. Formal solution

We first transform the initial-value problem of the preceding section to a boundary-value problem through the Laplace transformation

$$X(r,z,s) = \mathscr{L}\chi = \int_0^\infty e^{-st}\chi(r,z,t)\,dt. \tag{3.1}$$

Transforming (2.3)–(2.5) and (2.7), invoking the initial conditions (2.8) and (2.9), eliminating $\mathscr{L}\zeta$ between the transforms of (2.7 *a, b*), writing out $\nabla^2 X$, and invoking axial symmetry, we obtain

$$X_{rr} + r^{-1}X_r + \lambda^2 X_{zz} = 0 \tag{3.2}$$

and

$$(s^2/g)\,X + X_z - \lambda^{-2}z_0'(r)\,X_r = s\zeta_0(r) \quad [z = z_0(r)], \tag{3.3}$$

where

$$\lambda^2 = 1 + (2\omega/s)^2. \tag{3.4}$$

We have carried our formulation to this point in polar co-ordinates because of their inherent simplicity, but to expedite the solution of the boundary-value problem posed by (3.2) and (3.3) we require a co-ordinate system in which (*a*) the differential equation is separable and (*b*) the boundary $z = z_0(r)$ is a co-ordinate surface; (*a*) holds for polar co-ordinates, but (*b*) does not. Observing that (3.2) is Laplace's equation in the co-ordinates r and z/λ and that $z = z_0 = r^2/2l$ is a paraboloid of revolution, we are naturally led to introduce parabolic co-ordinates, say ξ and η, in an $(r, z/\lambda)$-space. Normalizing these co-ordinates such that $\xi = r$ and $\eta = 1$ on $z = z_0(r)$, we may pose the required transformation in the form

$$r = \xi\eta, \quad z - z_0(r) = \tfrac{1}{2}[l^{-1}\xi^2 + \lambda^2 l(1-\eta^2)] \quad (0 \leqslant \xi < \infty,\ 1 \leqslant \eta < \infty). \tag{3.5 a, b}$$

The differential metric is given by

$$(dr)^2 + \left(\frac{dz}{\lambda}\right)^2 = \left[\frac{\xi^2}{\lambda l} + \lambda l \eta^2\right]\left[\frac{(d\xi)^2}{\lambda l} + \lambda l (d\eta)^2\right]. \tag{3.6}$$

Transforming (3.2) and (3.3), we obtain

$$(\lambda l)^2\,\xi^{-1}(\xi X_\xi)_\xi + \eta^{-1}(\eta X_\eta)_\eta = 0 \tag{3.7}$$

and

$$(s^2/g)\,X - (\lambda^2 l)^{-1}X_\eta = s\zeta_0(\xi) \quad (\eta = 1). \tag{3.8}$$

We also have the finiteness conditions

$$|X| < \infty \quad (\xi \to 0, \infty;\ \eta \to \infty). \tag{3.9}$$

We remark that: ξ is a length and η is dimensionless; ξ and η are orthogonal co-ordinates in an $(r, z/\lambda)$-space but not in an (r, z)-space; ξ and η are real only for real values of λ^2; the differential equation (3.7) is elliptic for $\lambda^2 > 0$ and hyperbolic for $\lambda^2 < 0$, corresponding to waves of the first and second class.

A particular solution of (3.7) that satisfies the finiteness conditions (3.9) is given by

$$X_k = J_0(k\xi)\,K_0(\lambda k l \eta) \quad (\mathscr{R}\{\lambda\} > 0), \tag{3.10}$$

where J_0 is an ordinary Bessel function and K_0 a modified Bessel function of the second kind. We therefore may pose a general solution to (3.7) and (3.9) in the form (introducing a factor of g in order to simplify the subsequent expressions for the free-surface displacement)

$$X = g\int_0^\infty Z(k,s)\,J_0(k\xi)\,N(\lambda k l\eta)\,k\,dk, \tag{3.11}$$

where

$$N(\kappa\eta) = K_0(\kappa\eta)/K_0(\kappa), \quad \kappa = \lambda k l. \tag{3.12 a, b}$$

We may identify $Z(k,s)$ as the joint, Hankel–Laplace transform of the free-surface displacement in the limit $\eta \to 1+$ [see (3.17) below]. Substituting (3.11) into (3.8) and introducing the auxiliary function

$$\phi(\kappa) = K_1(\kappa)/K_0(\kappa) = iH_1^{(1)}(i\kappa)/H_0^{(1)}(i\kappa) \tag{3.13 a, b}$$

$$\sim 1+\tfrac{1}{2}\kappa^{-1}+O(\kappa^{-2}) \quad (\kappa\to\infty), \tag{3.13 c}$$

we obtain

$$\int_0^\infty [s^2+(gk/\lambda)\,\phi(\lambda k l)]\,Z(k,s)\,J_0(k\xi)\,k\,dk = s\zeta_0(\xi). \tag{3.14}$$

Substituting ζ_0 from (2.2) into (3.14) and then applying the Hankel inversion theorem, we obtain

$$Z(k,s) = As[s^2+(gk/\lambda)\,\phi(\lambda k l)]^{-1}. \tag{3.15}$$

Substituting (3.15) into (3.11) and taking the inverse Laplace transformation of the result, we obtain the formal solution

$$\chi(r,z,t) = Ag\mathscr{L}^{-1}s\int_0^\infty \frac{J_0(k\xi)\,N(\lambda k l\eta)\,k\,dk}{s^2+(gk/\lambda)\,\phi(\lambda k l)}. \tag{3.16}$$

The free-surface displacement is given by (2.7 *a*) as

$$\zeta(r,t) = g^{-1}\chi(r,z_0(r),t) \tag{3.17 a}$$

$$= \lim_{\eta\to 1+}\mathscr{L}^{-1}\int_0^\infty Z(k,s)\,J_0(k\xi)\,N(\lambda k l\eta)\,k\,dk \tag{3.17 b}$$

$$\equiv \mathscr{L}^{-1}\int_0^\infty Z(k,s)\,J_0(kr)\,N_1\,k\,dk, \tag{3.17 c}$$

where the symbol N_1 implies a passage to the limit $\eta = 1+$ *after* the double integral with respect to k and s has been either evaluated explicitly or rendered convergent for $\eta = 1$. We note that

$$N(\kappa\eta) \sim e^{-\kappa(\eta-1)}\,(|\kappa|\to\infty, \quad \mathscr{R}\{\kappa\}>0,\ \eta>1). \tag{3.18}$$

We may guarantee the restriction $\mathscr{R}\{\lambda\} > 0$, imposed in (3.10), by choosing that branch of λ [see (3.4)] which takes the value 1 at $s = \infty$ and drawing a branch cut along $\mathscr{R}\{s\}$ between the branch points at $s = \pm 2i\omega$. We show in the Appendix that the only other singularities of Z, *qua* function of s, are two poles on the imaginary axis outside of the branch cut, say $s = \pm i\sigma$, $\sigma > 2\omega$. Substituting λ from (3.4) into (3.15) and equating the denominator of the result to zero, we obtain the implicit dispersion equation

$$\sigma^2 = 4\omega^2 + gk\lambda\phi(kl\lambda), \quad \lambda = [1-(2\omega/\sigma)^2]^{\frac{1}{2}}. \tag{3.19a, b}$$

We remark that (3.19) has the explicit solution [cf. (1.6)]

$$\sigma^2 = [(gk)^2 + (2\omega^2)^2]^{\frac{1}{2}} + 2\omega^2 \quad (\alpha = \infty) \tag{3.20a}$$

$$= gk + 2\omega^2 + O(\omega^4/gk) \qquad (\alpha = \infty,\ \omega^2 \to 0). \tag{3.20b}$$

If, on the other hand, $\alpha = O(1)$, we must include the first two terms in the asymptotic approximation (A 13) to obtain

$$\sigma^2 = gk + 2\omega^2 + \tfrac{1}{2}(g/l) + O(\omega^4/gk) \quad [\alpha = O(1),\ \omega^2 \to 0]. \tag{3.21}$$

4. Free-surface displacement

We shall consider further only the free-surface displacement ζ. Substituting Z from (3.15) into (3.17c) and introducing the dimensionless displacement f according to (1.5), we obtain

$$f = r^2 \mathscr{L}^{-1}\left\{s\int_0^\infty \frac{J_0(kr)\,N_1\,k\,dk}{s^2 + (gk/\lambda)\,\phi(\lambda kl)}\right\}, \tag{4.1}$$

where

$$\mathscr{L}^{-1}\{\ \} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\{\ \}\,e^{st}\,ds, \tag{4.2}$$

and the path of integration in the s-plane passes to the right of both the poles at $s = \pm i\sigma$ and the branch cut between $s = \pm 2i\omega$.

It is, in principle, possible to develop f as a power series in t by expanding its Laplace transform about $s = \infty$. The result is far more cumbersome than in the classical problem, however, and the resulting representation does not lend itself to a wave interpretation. Accordingly, we note only the first approximation

$$f = r^2\mathscr{L}^{-1}\int_0^\infty J_0(kr)\,\{s^{-1} - gk\phi(kl)\,s^{-3} + O[(gk)^2\,\phi^2(kl)\,s^{-5},\ \omega^2 gk\phi(kl)\,s^{-5}]\}\,N_1\,k\,dk \quad (s \to \infty). \tag{4.3}$$

Introducing the dimensionless variables of (1.6) and the change of variable $kr = v$ and then inverting term by term, we obtain

$$f = \int_0^\infty J_0(v)\,\{-\tfrac{1}{2}\mu v\phi(v/\theta) + O[\mu^2 v^2\phi^2(v/\theta),\ \mu\tau^2 v\phi(v/\theta)]\}\,N_1\,v\,dv \quad (\mu \to 0). \tag{4.4}$$

Introducing the asymptotic expansion (3.13c), we obtain

$$f = \tfrac{1}{2}\mu[1 + O(\theta^2)] \quad (\mu \to 0,\ \theta \to 0). \tag{4.5}$$

If t is not sufficiently small to render a power-series approximation to f efficient, it becomes expedient to separate the solution into waves of the first and second class. Proceeding in the usual manner, we deform the path of integration of (4.2) into three closed loops, separately encircling the two poles and the branch cut. Referring to the discussion in § 1 above, we may identify the joint contribution of the poles as f_1 and the contribution of the branch cut as f_2.

Considering f_1 first, we introduce the change of variable $kr = u/\lambda$ and the dimensionless variables of (1.6) into (3.19 a, b) and (4.1) and then apply Cauchy's residue theorem to obtain

$$f_1(\mu, \theta; \alpha) = \int_0^\infty \cos(\sigma t)\, J_0(u/\lambda)\, N_1 u\, du \quad (4.6\,a)$$

$$= \int_0^\infty \cos\{\mu^{\frac{1}{2}}[u\phi(u/\theta) + 4\alpha\theta]^{\frac{1}{2}}\}\, J_0\{u[1 + 4\alpha\theta u^{-1}\phi^{-1}(u/\theta)]^{\frac{1}{2}}\}\, N_1 u\, du, \quad (4.6)$$

where

$$\sigma^2(u) = 4\omega^2 + (g/l\theta)\, u\phi(u/\theta). \quad (4.7)$$

Turning to f_2, we introduce the change of variable

$$s = 2i\omega \sin w, \quad \lambda = -i\cot w \quad (\mathscr{I}\{w\} < 0) \quad (4.8\,a, b)$$

in (4.2) we obtain

$$f_2 = \frac{2\alpha r^2}{\pi} \int_0^{2\pi} \exp\{2i\tau \sin w\} \cos^2 w\, dw \int_0^\infty \frac{J_0(kr)\, N_1 k\, dk}{kl\phi(-ikl\cot w) + 4i\alpha \sin w \cos w}$$

$$(\mathscr{I}\{w\} \to 0-). \quad (4.9)$$

Transforming the contributions of each of the four w-quadrants to the first quadrant, introducing the change of variable $kr = v$, and setting $\mathscr{I}\{w\} = 0-$ and $N_1 = 1$, we may reduce (4.9) to

$$f_2 = \frac{8\alpha\theta}{\pi} \mathscr{R} \int_0^{\frac{1}{2}\pi} \cos(2\tau \sin w) \cos^2 w\, dw \times \int_0^\infty \frac{J_0(v)\, v\, dv}{v\phi[-i(v/\theta)\cot w] + 4i\alpha\theta \sin w \cos w}. \quad (4.10)$$

Substituting ϕ from (3.13 b) into (4.10), taking the imaginary part of the result, and simplifying with the aid of the Wronskian of J_0 and Y_0, we obtain

$$f_2(\tau, \theta; \alpha) = \frac{8\alpha\theta}{\pi} \int_0^{\frac{1}{2}\pi} \cos(2\tau \sin w) \cos^2 w\, dw \int_0^\infty J_0(v)\, H\left(\frac{v}{\theta}\cot w, \cos^2 w\right) dv, \quad (4.11)$$

where

$$H(x, y) = (2x/\pi)\, |xH_1^{(1)}(x) + 4\alpha y H_0^{(1)}(x)|^{-2} \quad (4.12\,a)$$

$$= O[x(1 - 4\alpha y \log x)^{-2}] \quad (x \to 0) \quad (4.12\,b)$$

$$= 1 + O(x^{-2}) \quad (x \to \infty). \quad (4.12\,c)$$

Further progress appears to be possible only on the basis of approximate methods. Having already dealt briefly with power-series approximations, we shall devote the following section to asymptotic approximations.

5. Asymptotic approximation

We consider first the asymptotic approximation of f_1 for

$$\mu \gg 1,\ \theta,\ \alpha\theta. \quad (5.1)$$

Appealing to the fact, established in the Append'x, that $\kappa\phi(\kappa)$ is a monotonically increasing function of κ, we then may show that the integrand of (4.6 b) can have a point of stationary phase only for $u = O(\mu)$ and that the neighbourhood of such a point makes a contribution of $O(\mu)$ to the integral. We also may show, by Fourier-integral techniques, that the contributions of other neighbourhoods are at most $O(\mu^{-1})$. It follows that we may obtain the dominant terms in f_1 as $\mu\to\infty$ through a stationary-phase approximation based on the asymptotic approximation of $\phi(u/\theta)$. Substituting (3.13*c*) into (4.6*b*) and adopting the convention that $O(\theta^2)$ implies the largest of $O(\theta^2, \alpha\theta^2, \alpha^2\theta^2)$, we obtain

$$f_1(\mu,\theta;\alpha) \sim \int_0^\infty \cos\{\mu^{\frac{1}{2}}[u^{\frac{1}{2}}+(2\alpha+\tfrac{1}{4})\,\theta u^{-\frac{1}{2}}+O(\theta^2u^{-\frac{3}{2}})]\} \times J_0[u+2\alpha\theta+O(\theta^2u^{-1})]\,N_1\,u\,du \quad (\mu\to\infty). \tag{5.2}$$

We may now proceed as in the classical problem (Lamb, § 255). Replacing J_0 by its asymptotic expansion, replacing the resulting product of cosines by the sum of the cosines of the sum and difference of the two phases, and neglecting the cosine of the sum as having no point of stationary phase, we obtain

$$f_1 \sim (2\pi)^{-\frac{1}{2}}\int_0^\infty [u^{\frac{1}{2}}-\alpha\theta u^{-\frac{1}{2}}+O(u^{-\frac{3}{2}},\theta^2u^{-\frac{3}{2}})]\cos\{\mu^{\frac{1}{2}}[u^{\frac{1}{2}}+(2\alpha+\tfrac{1}{4})\,\theta u^{-\frac{1}{2}}]-u-2\alpha\theta + \tfrac{1}{4}\pi+O(u^{-1},\theta^2u^{-1},\mu^{\frac{1}{2}}\theta^2u^{-\frac{3}{2}})\}\,N_1\,du \quad (\mu\to\infty). \tag{5.3}$$

The point of stationary phase for (5.3) lies at

$$u_s = \tfrac{1}{2}\mu-(4\alpha+\tfrac{1}{2})\,\theta+O(\mu^{-1},\theta^2\mu^{-1}). \tag{5.4}$$

Evaluating the contribution of the neighbourhood of $u=u_s$ to the integral in the usual way and then taking the limit $\eta\to 1+$ [$N_1\to 1$; cf. (3.18)], we obtain

$$f_1(\mu,\theta;\alpha) \sim 2^{-\frac{3}{2}}[\mu-12\alpha\theta-\theta+O(\mu^0,\theta^2\mu^{-1})]\cos[\tfrac{1}{4}\mu+2\alpha\theta+\tfrac{1}{2}\theta+O(\mu^{-1},\theta^2\mu^{-1})] \quad (\mu\to\infty). \tag{5.5}$$

We remark that the approximation to the phase is more accurate than that to the amplitude.

Turning to f_2 and defining $f_2^{(0)}$ as that approximation to f_2 obtained by replacing H by its asymptotic value of 1 in (4.11) and $\Delta = f_2 - f_2^{(0)}$ as the corresponding error, we obtain

$$f_2^{(0)} = \frac{8\alpha\theta}{\pi}\int_0^{\frac{1}{2}\pi}\cos(2\tau\sin w)\cos^2 w\,dw\int_0^\infty J_0(v)\,dv \tag{5.6a}$$

$$= 2\alpha\theta\tau^{-1}J_1(2\tau) \tag{5.6b}$$

and

$$\Delta = \frac{8\alpha\theta}{\pi}\int_0^{\frac{1}{2}\pi}\cos(2\tau\sin w)\cos^2 w\,F(\theta\tan w,\,\cos^2 w)\,dw, \tag{5.7}$$

where

$$F(u,y) = \int_0^\infty J_0(v)\,[H(v/u,y)-1]\,dv \tag{5.8a}$$

$$= u\int_0^\infty J_0(xu)\,[H(x,y)-1]\,dx. \tag{5.8b}$$

To estimate Δ, we start from the known bounds

$$|J_0(v)| < 1 \tag{5.9a}$$

and

$$|J_0(v)| < |J_0^2(v)+Y_0^2(v)| < (2/\pi v)^{\frac{1}{2}}, \tag{5.9b}$$

by virtue of which (5.8 *b*) yields

$$|F(u,y)| < u\int_0^\infty |H(x,y)-1|\,dx = O(u) \quad (u\to 0) \tag{5.10 a}$$

and
$$|F(u,y)| < (2u/\pi)^{\frac{1}{2}}\int_0^\infty x^{\frac{1}{2}}\,|H(x,y)-1|\,dx = O(u^{\frac{1}{2}}) \quad (u\to\infty). \tag{5.10 b}$$

It follows that the integrand in (5.7) vanishes like $\theta w\cos(2\tau w)$ as $w\to 0$ and like $\theta^{\frac{1}{2}}\cos(2\tau)\,(\frac{1}{4}\pi - w)^{\frac{3}{2}}$ as $w\to\frac{1}{4}\pi$ and hence that

$$\Delta = O(\alpha\theta^2) \quad (\theta\to 0). \tag{5.11}$$

The estimate (5.11) is inadequate for $\tau \gg 1$ in consequence of the point of stationary phase at $w = \frac{1}{4}\pi$ in (5.7). To estimate Δ as $\tau\to\infty$, we integrate (5.7) by parts, introduce the change of variable $w = \frac{1}{4}\pi - \tau^{-\frac{1}{2}}\varphi$, and then use the bound (5.10 *b*) and the method of stationary phase to obtain

$$\Delta = -\frac{4\alpha\theta}{\pi\tau}\int_0^{\frac{1}{2}\pi}\sin(2\tau\sin w)\frac{d}{dw}[\cos wF(\theta\tan w,\,\cos^2 w)]\,dw \tag{5.12 a}$$

$$\sim -\frac{4}{\pi}\alpha\theta\tau^{-\frac{3}{2}}\int_0^\infty \sin(2\tau-\varphi^2)\frac{d}{d\varphi}\left[\varphi F\left(\frac{\theta\tau^{\frac{1}{2}}}{\varphi},\,0\right)\right]d\varphi \tag{5.12 b}$$

$$= O\left[\alpha\theta^{\frac{3}{2}}\tau^{-\frac{5}{4}}\int_0^\infty \sin(2\tau-\varphi^2)\,\varphi^{-\frac{1}{2}}\,d\varphi\right] \tag{5.12 c}$$

$$= O(\alpha\theta^{\frac{3}{2}}\tau^{-\frac{5}{4}}) \quad (\tau\to\infty). \tag{5.12 d}$$

Summing up, we have

$$f_2(\tau,\theta;\,\alpha) = 2\alpha\theta[\tau^{-1}J_1(2\tau) + O(\theta)] \quad (\theta\to 0) \tag{5.13 a}$$

$$= 2\pi^{-\frac{1}{2}}\alpha\theta\tau^{-\frac{3}{2}}[\sin(2\tau-\tfrac{1}{4}\pi) + O(\theta^{\frac{1}{2}}\tau^{\frac{1}{4}})] \quad (\tau\to\infty) \tag{5.13 b}$$

$$= O(\alpha\theta^{\frac{3}{2}}\tau^{-\frac{5}{4}}) \quad (\tau \gg \theta^{-2} \gg 1). \tag{5.13 c}$$

The implications of the results obtained in this section have been discussed in § 1 above.

6. Convex free-surface

We now proceed to sketch the required changes in the foregoing analysis for convex free surfaces, corresponding to $l < 0$ in (1.1). Of special interest for geophysical problems is the spherical free surface

$$z_0(r) = -R + (R^2 - r^2)^{\frac{1}{2}} \tag{6.1 a}$$

$$= -(r^2/2R)\,[1 + O(r^2/R^2)]. \tag{6.1 b}$$

Comparing (6.1 *b*) and (1.1) and setting $l = -R$, we may approximate (6.1 *a*) by (1.1) for $(r/R)^2 \ll 1$.

We first observe that the formulation of § 2, and also of § 3 through (3.4), is valid as it stands for $l < 0$ or, indeed, for any axisymmetric free surface. The transformation of the boundary-value problem to parabolic co-ordinates, (3.5)–(3.9), may be rendered valid for $l < 0$ simply by choosing the η-range $[0, 1]$, rather than $[1, \infty]$. The finiteness condition (3.9) then must be satisfied for $\eta = 0$, rather than $\eta \to \infty$; accordingly, we must replace K_0 by I_0, the modified Bessel function of the first kind, in (3.10)–(3.12) and ϕ by (note that $I_0' = I_1$, whereas $K_0' = -K_1$)

$$\phi^{(-)}(\kappa) = -I_1(\kappa)/I_0(\kappa) \tag{6.2}$$

in (3.14) *et seq.* We emphasize that $\kappa < 0$ for $l < 0$ if λ is real. We therefore are led to the solution of (3.17) with

$$Z(k,s) = As[s^2 + (gk/\lambda)\,\phi^{(-)}(\lambda kl)]^{-1} \tag{6.3a}$$

$$= As[s^2 + (g/\lambda^2\,|l|)\,\phi_1^{(-)}(\lambda kl)]^{-1}, \tag{6.3b}$$

where

$$\phi_1^{(-)}(\kappa) = \kappa I_1(\kappa)/I_0(\kappa). \tag{6.4}$$

Remarking that $\phi_1^{(-)}$, and hence also Z, is a single-valued function of s, we may proceed along the lines of the Appendix to prove that Z is a meromorphic function of s with an infinite set of poles confined to the imaginary axis, say $s = \pm i\sigma_0, \pm i\sigma_1, \pm i\sigma_2, \ldots$, where

$$\sigma_0 \gtreqless 2\omega \quad \text{as} \quad k \gtreqless 2\omega(\tfrac{1}{4}g\,|l|)^{-\frac{1}{2}} \tag{6.5a}$$

and

$$2\omega > \sigma_1 > \sigma_2 > \ldots. \tag{6.5b}$$

It follows that: (*a*) the two waves of the first class go over to waves of the second class for $k < 2\omega(\frac{1}{4}g\,|l|)^{-\frac{1}{2}}$; (*b*) the waves of the second class form a discrete, infinite spectrum with a limit point at $\sigma = 0$,† in remarkable contrast to the continuous spectrum $(-2\omega, 2\omega)$ that obtains for $l > 0$.

We may suppress the essential differences between the frequency spectra for concave and convex free surfaces of small curvature simply by replacing Z by its asymptotic approximation prior to the deformation of the path for the Laplace-transform inversion integral. We then have

$$Z \sim As\left[s^2 + \frac{gk}{\lambda} + \frac{1}{2}\frac{g}{\lambda^2 l} + O\left(\frac{g}{\lambda^3 kl^2}\right)\right] \quad (|l| \to \infty,\ \mathscr{R}\{\lambda\} > 0), \tag{6.6}$$

independently of sgn l. Substituting (6.6) into (3.17*c*) and then proceeding as in §§ 4 and 5, we obtain the results (4.5), (5.5), and (5.13*a*). We remark that the approximations (4.5) and (5.13*a*) are independent of sgn l, as also are the terms in $\alpha\theta$ in (5.5), whereas θ changes sign with l in (5.5). The argument is heuristic, to be sure, since (6.6) is not uniformly valid near either $k = 0$ or $\lambda = 0$. More rigorous derivations of (4.5), (5.5), and (5.13*a*) for $l < 0$ may be constructed along the lines followed in §§ 4 and 5 but do not appear to be worth presenting here. Estimating the error Δ, as in (5.7)–(5.12), would appear to be more difficult, and the error terms given in (5.13*b*, *c*) may not be valid for $l < 0$.

Appendix

We wish to prove that there are two, and only two, poles of Z, *qua* function of s, that these poles lie on the imaginary axis—say $s = \pm i\sigma$—and that $\sigma > 2\omega$.

We begin by considering the denominator of $Z(k,s)$, (3.15), as a function of $\kappa = \lambda kl$. Solving (3.4) for s^2, we obtain

$$s^2 = -4\omega^2[1-(\kappa/\kappa_1)^2]^{-1} \quad (\kappa_1 = kl). \tag{A 1}$$

† This would present no essential difficulty in effecting the inverse Laplace transformation, since the s-plane could be mapped on a $(1/s)$-plane. The result is a series converging like n^{-2}.

Substituting (A 1) into the denominator of (3.15), we may place the result in the form

$$Z = \frac{As}{\omega^2}\left(\frac{\kappa}{\kappa_1}\right)^2 [\phi_1(\kappa)-\psi(\kappa)]^{-1}, \tag{A 2}$$

where

$$\phi_1(\kappa) = \frac{\kappa K_1(\kappa)}{K_0(\kappa)} = \frac{i\kappa H_1^{(1)}(i\kappa)}{H_0^{(1)}(i\kappa)} = \kappa\phi(\kappa) \tag{A 3a, b}$$

and

$$\psi(\kappa) = 4\kappa^2/(\kappa_1^2-\kappa^2). \tag{A 4}$$

The poles of Z then correspond to the zeros of $\phi_1(\kappa)-\psi(\kappa)$ in $|\kappa| > 0$; $\kappa = 0$ obviously is not a pole of Z. We remark that the imaginary part of ϕ_1 is given by $-(2/\pi)\,|H_0^{(1)}(i\kappa)|^{-2}$ for imaginary values of κ, in consequence of which $\phi_1-\psi$ cannot have imaginary zeros. Similarly, $\phi_1-\psi$ cannot have negative real zeros.

We next establish that Z can have poles only for $0 < \kappa < \kappa_1$. Multiplying the differential equation

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}-\kappa^2\right) K_0(\kappa z) = 0 \tag{A 5}$$

through by $zK_0^*(\kappa^* z)$, where K_0^* is the complex conjugate of K_0, integrating by parts between $z = 1$ and $z = \infty$, dividing the result through by $K_0 K_0^*$, and substituting ϕ_1 from (A 3), we obtain

$$\phi_1(\kappa) = \frac{\kappa^2}{|K_0(\kappa)|^2}\int_0^\infty [(\kappa^*/\kappa)\,|K_1(\kappa z)|^2+|K_0(\kappa z)|^2]\,z\,dz. \tag{A 6}$$

Equating the real and imaginary parts of (A 4) and (A 6), we infer that either $\kappa = 0$ or $\kappa_1^2-\kappa^2 > 0$. Having already established that $\phi_1-\psi$ cannot have either imaginary or negative real zeros and that $\kappa = 0$ is trivial, we conclude that the non-trivial zeros of $\phi_1-\psi$ lie in $0 < \kappa < \kappa_1$.

We now proceed to show that $\phi_1(\kappa)$ is a monotonically increasing function in $\kappa \geqslant 0$ that intersects $\psi(\kappa)$ once and only once. We first remark that, from (A 3) and (A 5), ϕ_1 satisfies the Riccati equation

$$\phi\kappa_1'(\kappa) = \phi_1^2-\kappa^2. \tag{A 7}$$

Starting from (A 3a) and the known integral of $zK_0^2(z)$, we also have

$$\phi_1^2-\kappa^2 = \frac{2\kappa^2}{K_0^2(\kappa)}\int_1^\infty K_0^2(\kappa x)\,x\,dx > 0. \tag{A 8}$$

It follows that $\phi_1'(\kappa) > 0$ for $\kappa > 0$. Comparing (A 6) and (A 8) and noting, from the integral representation

$$K_1(\kappa)-K_0(\kappa) = \int_0^\infty \exp\{-\kappa\cosh t\}\,(\cosh t-1)\,dt, \tag{A 9}$$

that $K_1 > K_0$ for $\kappa > 0$, we find that

$$\kappa\phi_1' = \phi_1^2-\kappa^2 < \phi_1. \tag{A 10}$$

It follows that

$$\phi_1'' < (\kappa^{-1}\phi_1)' = \kappa^{-2}(\kappa\phi_1'-\phi_1) < 0 \tag{A 11}$$

and hence that $\phi_1(\kappa)$ is concave downwards in $\kappa > 0$.

Differentiating (A 4), we find that $\psi(\kappa)$ is monotonically increasing and concave upwards in $(0, \kappa_1)$. Finally, we remark that $\psi < \phi_1$ for sufficiently small, positive κ (since ψ and ϕ_1 tend to zero like κ^2 and $-1/\log \kappa$, respectively), and that $\psi \to \infty$ as $\kappa \to \kappa_1$. It follows that $\phi_1 = \psi$ once, and only once, in $(0, \kappa_1)$. Referring to (A 1) and the choice of branch cut already discussed in § 3, we conclude that there are two, and only two, poles of $Z(k, s)$ and that these poles lie on the imaginary axis in the s-plane and exceed 2ω in magnitude.

Another consequence of (A 7) and (A 10) is

$$\kappa < \phi_1(\kappa) < (\kappa^2 + \tfrac{1}{4})^{\frac{1}{2}} + \tfrac{1}{2} \quad (\kappa > 0). \tag{A 12}$$

An interpolation between these bounds that agrees with the first two terms in the asymptotic expansion of ϕ_1 is given by

$$\phi_1(\kappa) \doteqdot \kappa + \tfrac{1}{4} \quad (\kappa > 0). \tag{A 13}$$

We have used (A 13) in the asymptotic approximations of § 4, where it is justified directly by $\kappa \gg 1$, but it seems likely that it should be a good approximation for only moderately large values of κ.

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